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ANTIPLANE STRAIN IN A NONLINEARLY ELASTIC INCOMPRESSIBLE BODY

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The stress–strain state of an incompressible cylindrical elastic body with antiplane strain under the action of potential forces and surface loading constant along the body is considered in a nonlinear formulation in actual variables. The stresses are expressed via the pressure and independent strains, the pressure is expressed via the force and elastic potentials, and nonlinear boundary-value problems are posed for strains (and displacements). Various methods for solving these problems are developed. For the nonlinear equations obtained, some analytical solutions containing free parameters are given, which can be used as a basis for solving particular problems.

Key words: *stresses, strains, displacement, potential, nonlinear problem, Legendre transform, strain plane.*

Investigations of some important problems of elasticity show that it is necessary to take into account high strains of the body and nonlinear behavior of the material. Under these conditions, the linear theory of elasticity does not ensure required accuracy, and the nonlinear theory, which allows for geometrical and physical nonlinearity, has to be used.

We consider antiplane strain in an isotropic cylindrical body under the action of potential forces on the basis of a nonlinear model of an incompressible body. The governing equations of this model are the equations of equilibrium, Murnaghan law, incompressibility condition, dependence of the elastic potential on the basis invariants of strain, and expression of invariants via the strain components and of strain components via the displacement. In the variables of the actual state x_1, x_2, and x_3 ($x_1 = x$ and $x_2 = y$ are the transverse coordinates and x_3 is the longitudinal coordinate), these relations have the form [1]

$$\frac{\partial}{\partial x_l}(P_{kl} - V\delta_{kl}) = 0, \qquad P_{kl} = -q_*\delta_{kl} + (\delta_{kn} - 2E_{kn})\frac{\partial U}{\partial E_{ln}}, \qquad U = U(E, E_2, E_3),$$

$$E = E_{nn}, \qquad 2E_2 = E_{nn}E_{mm} - E_{nm}E_{mn}, \qquad E_3 = |E_{kl}|, \tag{1}$$

$$2E_{kl} = \frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} - \frac{\partial u_n}{\partial x_k}\frac{\partial u_n}{\partial x_l}, \qquad E - 2E_2 + 4E_3 = 0,$$

where q_* is the Lagrangian factor, U and V are the elastic and force potentials, E, E_2, and E_3 are the strain invariants, u_k, P_{kl}, and E_{kl} are the components of the displacement, Cauchy stresses, and Almansi strains, and δ_{kl} is the Kronecker symbol (the subscripts take the values of 1, 2, and 3; summation is performed over repeated subscripts).

For a cylinder with a cross-sectional area S and a contour L subjected to antiplane strain along the body [2], we assume that the elastic and force potentials, the side load p_k, and the longitudinal component P_3 of the resultant end-face load are known:

$$u_1 = u_2 = 0, \qquad u_3 = w(x, y), \qquad U = U(E, E_2, E_3), \qquad V = V(x, y); \tag{2}$$

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$$p_k\Big|_L = p_k(x,y), \qquad P_3 = \int\limits_S p_3\,dS. \tag{3}$$

According to Eqs. (1) and (2), the strains are nonlinearly presented via the displacement (geometrical nonlinearity) and can be expressed in terms of two independent components E_{31} and E_{32} related by the differential compatibility condition

$$\frac{\partial E_{32}}{\partial x} - \frac{\partial E_{31}}{\partial y} = 0. \tag{4}$$

The strain invariants are nonpositive, are expressed via the linear invariant E, and satisfy the incompressibility condition (which justifies the use of the model of an incompressible body). The elastic potential is expressed via the linear invariant in the form $U = f(E)$; hence, the Murnaghan law predicts a quasi-linear dependence of stresses on strains (physical nonlinearity) and pressure q, which admits the presentation of stresses via the pressure and independent strains:

$$\begin{gathered}
P_{11} = -q + 4f'(E)E_{31}^2, \qquad P_{22} = -q + 4f'(E)E_{32}^2, \qquad P_{33} = -q,\\
P_{12} = 4f'(E)E_{31}E_{32}, \qquad P_{31} = -2f'(E)E_{31}, \qquad P_{32} = -2f'(E)E_{32},\\
q = q_* - f'(E), \qquad E = -2(E_{31}^2 + E_{32}^2).
\end{gathered} \tag{5}$$

The prime here denoted the derivative of the function $f(E)$ with respect to the argument.

By virtue of Eq. (1), the formulas $2E_{31} = \partial w/\partial x$ and $2E_{32} = \partial w/\partial y$ are valid, and the displacement is expressed via strains as

$$w = 2\int\limits_{(x_*,y_*)}^{(x,y)} (E_{31}\,dx + E_{32}\,dy) + w_*, \qquad w_* = w(x_*,y_*), \quad (x_*,y_*) \in L, \tag{6}$$

where the integral, by virtue of Eq. (4), is independent of the path of integration, and the additive constant is determined by the displacement prescribed at the boundary point. Thus, the stresses (5) and displacement (6) are determined by the pressure and independent strains.

In the first and second equations of equilibrium in (1), the pressure is expressed via the force and elastic potentials by the integral of equations where the integration constant is found from the integral condition in (3):

$$q = h - V - f, \qquad h = \frac{1}{S}\Big(\int (V+f)\,dS - P_3\Big). \tag{7}$$

In particular, if the axial component of the resultant end-face load equals zero, this constant equals the mean sum of the force and elastic potentials in the body cross section, and the pressure equals the deviation of the sum of the potentials from the mean value.

The independent strains are determined by a nonlinear system of equations consisting of the third equation of equilibrium in (1) and the compatibility condition of strains (4):

$$\begin{gathered}
\frac{\partial\,(f'E_{31})}{\partial x} + \frac{\partial\,(f'E_{32})}{\partial y} = 0, \qquad \frac{\partial E_{32}}{\partial x} - \frac{\partial E_{31}}{\partial y} = 0,\\
f' = f'(E), \qquad E = -2(E_{31}^2 + E_{32}^2).
\end{gathered} \tag{8}$$

We assume that the condition $f''/f' \leqslant 0$ sufficient for this system to be elliptical is satisfied; hence, a boundary-value problem with prescribed boundary strains is well-posed for system (8).

The boundary conditions in (3) written in the natural basis of the contour L (normal n, tangent t, and binormal b) [3]

$$\begin{gathered}
p_n = V + f - h + 4f'e_n^2, \qquad p_t = 4f'e_ne_t, \qquad p_b = -2f'e_n,\\
e_n = E_{31}n_1 + E_{32}n_2, \qquad e_t = -E_{31}n_2 + E_{32}n_1,\\
E = -2(e_n^2 + e_t^2) \qquad \text{on} \quad L
\end{gathered} \tag{9}$$

determine the restrictions on loading [first equality in (9)] and the boundary values of the independent strains

$$E_{31}\Big|_L = e_n n_1 - e_t n_2, \qquad E_{32}\Big|_L = e_n n_2 + e_t n_1, \tag{10}$$

$$e_t = p_t/(2p_b), \qquad p_b + 2e_n f'(E) = 0, \qquad E = -2e_n^2 - p_t^2/(2p_b^2).$$

It follows from relations (7), (8), and (10) that the potential forces affect the pressure and do not affect the strains. According to Eq. (5), therefore, the potential forces affect the extension–compression stresses and do not affect the shear stresses.

In the cylindrical coordinates s, v, and z (z varies along the body and s and v vary in the cross-sectional plane), the stresses are expressed via the pressure q and independent strains E_{zs} and E_{zv}:

$$\begin{gathered} P_{ss} = -q + 4f'E_{zs}^2, \qquad P_{vv} = -q + 4f'E_{zv}^2, \qquad P_{zz} = -q, \qquad P_{sv} = 4f'E_{zs}E_{zv}, \\ P_{zs} = -2f'E_{zs}, \qquad P_{zv} = -2f'E_{zv}, \qquad f' = f'(E), \qquad E = -2(E_{zs}^2 + E_{zv}^2). \end{gathered} \tag{11}$$

In Eqs. (11), the pressure is determined by Eq. (7), and the independent strains are the solution of the boundary-value problem

$$\begin{gathered} \frac{\partial(sf'E_{zs})}{\partial s} + \frac{\partial(f'E_{zv})}{\partial v} = 0, \qquad \frac{\partial(sE_{zv})}{\partial s} - \frac{\partial E_{zs}}{\partial v} = 0, \\ f' = f'(E), \qquad E = -2(E_{zs}^2 + E_{zv}^2), \\ E_{zs} = e_n n_s - e_t n_v, \qquad E_{zv} = e_n n_v + e_t n_s \qquad \text{on} \quad L, \end{gathered} \tag{12}$$

where e_n and e_t are defined in Eqs. (10). In some cases, this system admits analytical solutions.

If the strains depend only on the polar radius [$E_{zs}(s)$ and $E_{zv}(s)$], then, according to Eqs. (12), we also have the dependences $E(s)$ and $f'(s)$, and Eqs. (12) acquire the form

$$\frac{d(sf'E_{zs})}{ds} = 0, \qquad \frac{d(sE_{zv})}{ds} = 0.$$

For an arbitrary elastic potential, these equations have a solution with two arbitrary constants:

$$sE_{zv} = A, \qquad sf'E_{zs} = C, \qquad A = \text{const}, \qquad C = \text{const}. \tag{13}$$

For a quadratic Rivlin–Saunders potential modeling high elastic strains of rubber-like materials [2]

$$\begin{gathered} f = aE^2 - bE + c, \qquad a > 0, \quad b > 0, \quad c > 0, \quad E < 0, \\ f' = -b(1 + 4k(E_{zs}^2 + E_{zv}^2)), \qquad f'' = 2a, \qquad k = a/b, \end{gathered} \tag{14}$$

the component E_{zv} is determined by the first equality in (13), and the component E_{zs} is determined by a cubic equation, which has one real solution [4]:

$$\begin{gathered} E_{zs}^3 + mE_{zs} + h = 0, \quad m = \frac{s^2 + 4kA^2}{4ks^2}, \quad h = \frac{B}{4ks} \quad \Big(C = bB, \quad H = \frac{m^3}{27} + \frac{h^2}{4} > 0\Big), \\ E_{zs} = \sqrt[3]{-h/2 + \sqrt{H}} + \sqrt[3]{-h/2 - \sqrt{H}}, \qquad E_{zv} = A/s. \end{gathered} \tag{15}$$

The condition of ellipticity $f''/f' \leqslant 0$ is satisfied for the Rivlin–Saunders potential.

In the case of weak physical nonlinearity, with the coefficient at the quadratic term in potential (14) being significantly smaller than the coefficient at the linear term ($a/b = k \ll 1$), the independent strains (15) in the linear approximation in terms of k are

$$E_{zv} = \frac{A}{s}, \quad E_{zs} = -\frac{B}{s}\Big(1 - 4k\,\frac{A^2 + B^2}{s^2}\Big), \quad A = \text{const}, \quad B = \text{const}, \quad k \ll 1.$$

If the strains depend only on the polar angle [$E_{zs}(v)$ and $E_{zv}(v)$], system (12) also predicts the dependences $E(v)$ and $f'(v)$, and system (12) reduces to an equation for one component E_{zs}:

$$E_{zv} = \frac{dE_{zs}}{dv}, \quad f'E_{zs} + \frac{d}{dv}\Big(f'\,\frac{dE_{zs}}{dv}\Big) = 0, \quad f' = f'(E), \quad E = -2\Big(E_{zs}^2 + \Big(\frac{dE_{zs}}{dv}\Big)^2\Big).$$

For a quadratic elastic potential (14), this equation has the form

$$\Big[1+4k\Big(E_{zs}^2+3\Big(\frac{dE_{zs}}{dv}\Big)^2\Big)\Big]\Big(E_{zs}+\frac{d^2E_{zs}}{dv^2}\Big)=0$$

and reduces to vanishing of the second term. It follows from the form of the general solution of this equation that the independent strains are periodic functions of the polar angle with two arbitrary constants, which have the meaning of the amplitude and initial phase:

$$E_{zs}=C\sin{(v+D)},\qquad E_{zv}=C\cos{(v+D)},\qquad C=\text{const},\qquad D=\text{const}.$$

The system for strains (8) with allowance for Eq. (6) can be written in the form of one equation with respect to the longitudinal displacement $w=w(x,y)$:

$$\frac{\partial}{\partial x}\Big[f'(E)\,\frac{\partial w}{\partial x}\Big]+\frac{\partial}{\partial y}\Big[f'(E)\,\frac{\partial w}{\partial y}\Big]=0,\qquad E=-\frac{1}{2}\Big[\Big(\frac{\partial w}{\partial x}\Big)^2+\Big(\frac{\partial w}{\partial y}\Big)^2\Big]. \tag{16}$$

Together with displacement (6) at the boundary

$$w\Big|_L=2\int\limits_0^s\Big(E_{31}(s)x'(s)+E_{32}(s)y'(s)\Big)\,ds+w_*$$

Eq. (16) forms the boundary-value problem for the displacement. We present this nonlinear equation in the form of a linear equation [5, p. 46] by applying the Legendre transform to the function $w=w(x,y)$, i.e., we introduce new independent variables ξ and η and a new unknown function $\Phi=\Phi(\xi,\eta)$ by the formulas

$$\xi=\frac{\partial w}{\partial x},\qquad \eta=\frac{\partial w}{\partial y},\qquad \Phi=x\xi+y\eta-w. \tag{17}$$

We assume that the Jacobian of transformation (17)

$$\Delta\equiv\frac{D(\xi,\eta)}{D(x,y)}=\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}-\Big(\frac{\partial^2 w}{\partial x\,\partial y}\Big)^2$$

is not equal to zero. We use r and θ to denote the polar coordinates in the plane (ξ,η). Then the function $\Phi=\Phi(r,\theta)$ obeys the linear equation

$$g(r)r^2\,\frac{\partial^2\Phi}{\partial r^2}+r\,\frac{\partial\Phi}{\partial r}+\frac{\partial^2\Phi}{\partial\theta^2}=0, \tag{18}$$

where

$$g(r)=(1-r^2f''/f')^{-1},\qquad f=f(E),\qquad E=-r^2/2. \tag{19}$$

The condition $r^2f''/f'<1$ ensures ellipticity of Eq. (18). The formulas of the inverse transform also hold:

$$x=\frac{\partial\Phi}{\partial r}\,\cos\theta-\frac{1}{r}\,\frac{\partial\Phi}{\partial\theta}\,\sin\theta,\qquad y=\frac{\partial\Phi}{\partial r}\,\sin\theta+\frac{1}{r}\,\frac{\partial\Phi}{\partial\theta}\,\cos\theta,\qquad w=r\,\frac{\partial\Phi}{\partial r}-\Phi. \tag{20}$$

Formulas (20) for the solution $\Phi(r,\theta)$ of Eq. (18) determine the displacement in the physical plane (x,y) in the parametric form.

If we assume that

$$\Phi_m(r,\theta)=Z_m(r)\,\mathrm{e}^{im\lambda\theta}\qquad (i^2=-1,\quad m=0,\pm1,\pm2,\ldots,\quad \lambda=\text{const}) \tag{21}$$

in Eq. (18), then the function $Z_m(r)$ satisfies the equation

$$g(r)r^2\,\frac{d^2Z_m(r)}{dr^2}+r\,\frac{dZ_m(r)}{dr}-\lambda^2m^2Z_m(r)=0. \tag{22}$$

Knowing the solution of this equation and taking into account expression (21), we can construct classes of particular solutions of Eq. (18) of the form

$$\sum_m Z_m(r)(a_m\cos{(m\lambda\theta)}+b_m\sin{(m\lambda\theta)}),$$

where a_m and b_m are arbitrary constants.

Let us consider some examples.

Example 1. Let $\lambda = 1$, $m = 1$, and $f = A\ln(1-2\alpha E)$, where $A > 0$ and $\alpha > 0$ are constants, and $E < 0$. Then the ellipticity condition $r^2 f''/f' < 1$ acquires the form $\alpha r^2 < 1$. In this case, according to Eq. (19), we have

$$g(r) = \frac{1+\alpha r^2}{1-\alpha r^2}$$

and Eq. (22) for the function $Z_1(r) = Z(r)$ takes the form

$$(1+\alpha r^2)r^2\,\frac{d^2 Z(r)}{dr^2} + (1-\alpha r^2)r\,\frac{dZ(r)}{dr} - (1-\alpha r^2)Z(r) = 0. \qquad (23)$$

The particular solutions of this equation are the functions

$$Z_1 = r, \qquad Z_2 = 2\alpha r\ln r - 1/r,$$

hence, the general solution of Eq. (23) has the form

$$Z_m(r) = C_1 Z_1 + C_2 Z_2 = r[C_1 + C_2(2\alpha\ln r - 1/r^2)], \qquad C_1 = \text{const}, \quad C_2 = \text{const}.$$

Example 2. Let $f = AE^{2k} + B$, where A, B, and k are positive constants ($k = 1, 2, \dots$), $E < 0$. Then, we have $r^2 f''/f' = -2(2k-1)$, and the ellipticity condition $r^2 f''/f' < 1$ is satisfied. In this case, according to Eq. (19), we have

$$g(r) = (4k-1)^{-1} = n^{-1} \qquad (n = 4k-1),$$

and Eq. (22) acquires the form

$$r^2\,\frac{d^2 Z_m(r)}{dr^2} + nr\,\frac{dZ_m(r)}{dr} - n\lambda^2 m^2 Z_m(r) = 0.$$

The general solution of this equation is written as

$$Z_m(r) = C_1 r^{n_1} + C_2 r^{n_2},$$

$$n_1 = (1-n)/2 + \sqrt{(1-n)^2/4 + n\lambda^2 m^2}, \qquad n_2 = (1-n)/2 - \sqrt{(1-n)^2/4 + n\lambda^2 m^2},$$

$$C_1 = \text{const}, \qquad C_2 = \text{const}.$$

The equations for the independent strains (8) with a quadratic potential (14) can be also examined with the use of strain potentials. Let us pass from the strains E_{31} and E_{32} to the potentials (displacement w and strain function v) by the formulas

$$2E_{31}f' = -b\,\frac{\partial v}{\partial y}, \qquad 2E_{32}f' = b\,\frac{\partial v}{\partial x}; \qquad (24)$$

$$2E_{31} = \frac{\partial w}{\partial x}, \qquad 2E_{32} = \frac{\partial w}{\partial y}. \qquad (25)$$

Then, the first equation in system (8) is satisfied by virtue of Eq. (24), and the second equation is satisfied by virtue of Eq. (25). Eliminating strains from Eqs. (24) and (25), we obtain the nonlinear dependences between the potentials

$$\frac{\partial w}{\partial x} = \frac{1}{f'_*}\,\frac{\partial v}{\partial y}, \quad \frac{\partial w}{\partial y} = -\frac{1}{f'_*}\,\frac{\partial v}{\partial x}, \quad f'_* = -\frac{f'}{b} = 1 + k\Big[\Big(\frac{\partial w}{\partial x}\Big)^2 + \Big(\frac{\partial w}{\partial y}\Big)^2\Big]. \qquad (26)$$

With the use of differentiation, we can eliminate one potential from Eq. (26) and obtain a second-order differential equation for the second potential. In particular, the equation for the displacement has the form of Eq. (16).

In the complex variables of the plane (x, y)

$$z = x + iy, \qquad \bar z = x - iy, \qquad \partial_x = \partial_z + \partial_{\bar z}, \qquad \partial_y = i(\partial_z - \partial_{\bar z}),$$

relations (24)–(26) have the form

$$E^{31} = E_{31} + iE_{32} = g\,\mathrm{e}^{ih}, \qquad E^{31} f'_* = -i\,\frac{\partial v}{\partial \bar z}, \qquad E^{31} = \frac{\partial w}{\partial \bar z},$$

$$\frac{\partial w}{\partial \bar z} = -\frac{i}{f'_*}\,\frac{\partial v}{\partial \bar z}, \qquad f'_* = 1 + 4k\,\frac{\partial w}{\partial z}\,\frac{\partial w}{\partial \bar z} = 1 + 4kE^{31}\bar E^{31} = 1 + 4kg^2, \tag{27}$$

where g and h are the polar coordinates and E^{31} and $\bar E^{31}$ are the complex coordinates in the strain plane (E_{31}, E_{32}).

The nonlinearity of the fourth equation in (27) is caused by the quantity f'_* depending on the polar radius g. To obtain a linear equation, we pass from the variables z and $\bar z$ to the variables g and h (E^{31} and $\bar E^{31}$):

$$g = \sqrt{E^{31}\bar E^{31}}, \qquad h = \frac{1}{2i}\,\ln\frac{E^{31}}{\bar E^{31}}.$$

For this purpose, we assume that $w = w(z,\bar z)$ and $v = v(z,\bar z)$, use Eq. (27), and consider the expression

$$\frac{1}{2}\,dw + \frac{i}{2f'_*}\,dv = \frac{dz}{2}\Big(\frac{\partial w}{\partial z} + \frac{i}{f'_*}\,\frac{\partial v}{\partial z}\Big) + \frac{d\bar z}{2}\Big(\frac{\partial w}{\partial \bar z} + \frac{i}{f'_*}\,\frac{\partial v}{\partial \bar z}\Big) = \bar E^{31}\,dz = g\,\mathrm{e}^{-ih}\,dz,$$

which (for $g \neq 0$) yields

$$dz = \frac{\mathrm{e}^{ih}}{2g}\Big(dw + \frac{i}{f'_*}\,dv\Big). \tag{28}$$

Assuming further that w, v, and z are functions of g, h, we find from Eq. (28) that

$$\frac{\partial z}{\partial g} = \frac{\mathrm{e}^{ih}}{2g}\Big(\frac{\partial w}{\partial g} + \frac{i}{f'_*}\,\frac{\partial v}{\partial g}\Big), \qquad \frac{\partial z}{\partial h} = \frac{\mathrm{e}^{ih}}{2g}\Big(\frac{\partial w}{\partial h} + \frac{i}{f'_*}\,\frac{\partial v}{\partial h}\Big). \tag{29}$$

Eliminating z from these equalities, equating the mixed derivatives $\partial^2 z/\partial g\,\partial h$ and $\partial^2 z/\partial h\,\partial g$, separating the real and imaginary parts in the resultant equality, and taking into account Eq. (27), we obtain the linear equations

$$\frac{\partial w}{\partial h} = \frac{g}{1+4kg^2}\,\frac{\partial v}{\partial g}, \qquad \frac{\partial v}{\partial h} = -\frac{g(1+4kg^2)^2}{1+12kg^2}\,\frac{\partial w}{\partial g}. \tag{30}$$

In the case of weak physical nonlinearity ($k \ll 1$), the coefficients at the derivatives in the right sides of Eqs. (30) in the linear approximation in terms of k differ only by their signs:

$$\frac{\partial w}{\partial h} = g(1-4kg^2)\,\frac{\partial v}{\partial g}, \qquad \frac{\partial v}{\partial h} = -g(1-4kg^2)\,\frac{\partial w}{\partial g}. \tag{31}$$

By replacing the variable g by t

$$t = \frac{1}{2}\,\ln\frac{1-4kg^2}{g^2} \qquad [g = \mathrm{e}^{-3t}(\mathrm{e}^{2t} - 2k)], \tag{32}$$

we can transform Eqs. (31) to the Cauchy–Riemann equations

$$\frac{\partial w}{\partial t} = \frac{\partial v}{\partial h}, \qquad \frac{\partial w}{\partial h} = -\frac{\partial v}{\partial t}. \tag{33}$$

By introducing the complex function u of the complex variables Z and $\bar Z$ in the plane (t,h)

$$u = w(Z,\bar Z) + iv(Z,\bar Z), \qquad Z = t + ih, \quad \bar Z = t - ih,$$
$$2\,\frac{\partial}{\partial Z} = \frac{\partial}{\partial t} - i\,\frac{\partial}{\partial h}, \qquad 2\,\frac{\partial}{\partial \bar Z} = \frac{\partial}{\partial t} + i\,\frac{\partial}{\partial h}, \tag{34}$$

we can write Eqs. (33) in the complex form

$$\frac{\partial u}{\partial \bar Z} = 0.$$

By means of integration, we find u as an arbitrary function W of the variable Z (complex potential) and also w and v as the real and imaginary parts of the function W:

$$u = W(Z), \qquad w = \operatorname{Re} W, \qquad v = \operatorname{Im} W. \tag{35}$$

Setting the contour L by the equations $x = x(s)$ and $y = y(s)$ (s is the arc of L) and the boundary conditions for the strains $E_{31}(s)$ and $E_{32}(s)$ [see (10)] determines, according to Eq. (6), the boundary displacement

$$w(s) = 2\int_{s_*}^{s}\Big[E_{31}(s)x'(s) + E_{32}(s)y'(s)\Big]\,ds + w_*, \tag{36}$$

and also [according to Eqs. (27), (32), and (34)] the quantities $g(s)$, $h(s)$, $t(s)$, $Z(s)$, and $\bar{Z}(s)$. Using representation (35) of the displacement in terms of the complex potential and its value (36) at the domain boundary, we obtain a standard boundary-value problem for the potential [6]

$$\operatorname{Re} W(Z)\Big|_L = w(s). \tag{37}$$

The potential $W(Z)$ found from Eq. (37) determines displacement (35) as a function of Z and $\bar{Z}$: $w(Z,\bar{Z}) = (W(Z)+\bar{W}(\bar{Z}))/2$, which can be written in the variables z and $\bar{z}$. For this purpose, with allowance for Eqs. (34) and (35) and the relations

$$g = \mathrm{e}^{-t}(1-2k\,\mathrm{e}^{-2t}), \qquad f'_* = 1+4k\,\mathrm{e}^{-2t}, \qquad t = \frac{Z+\bar{Z}}{2}, \qquad h = \frac{Z-\bar{Z}}{2i} \qquad (k \ll 1),$$

we present equality (28) in the form

$$dz = (1/2)\,\mathrm{e}^{t+ih}(1+2k\,\mathrm{e}^{-2t})[(1-2k\,\mathrm{e}^{-2t})\,dW + 2k\,\mathrm{e}^{-2t}\,d\bar{W}]$$
$$= (1/2)\,\mathrm{e}^{Z}\,W'(Z)\,dZ + k\,\mathrm{e}^{-\bar{Z}}\,\bar{W}'(\bar{Z})\,d\bar{Z}.$$

As a result of integration, we obtain

$$z = \frac{1}{2}\int \mathrm{e}^{Z}\,W'(Z)\,dZ + k\int \mathrm{e}^{-\bar{Z}}\,\bar{W}'(\bar{Z})\,d\bar{Z} + D, \qquad D = \text{const}. \tag{38}$$

Adding a complex-conjugate equality to (38), we obtain the dependences

$$z = z(Z,\bar{Z}), \qquad \bar{z} = \bar{z}(Z,\bar{Z}). \tag{39}$$

The Jacobian of this transformation calculated with allowance for relations (32) and (34) differs from zero

$$\frac{\partial(z,\bar{z})}{\partial(Z,\bar{Z})} = \frac{\partial(z,\bar{z})}{\partial(g,h)}\,\frac{\partial(g,h)}{\partial(t,h)}\,\frac{\partial(t,h)}{\partial(Z,\bar{Z})} = \frac{\mathrm{e}^{-t}}{2i(1+6k\,\mathrm{e}^{-2t})}\,\frac{\partial(z,\bar{z})}{\partial(g,h)} \neq 0$$

because, according to Eqs. (29), (30), and (27), we have

$$\frac{\partial(z,\bar{z})}{\partial(g,h)} = \frac{i}{2g^2 f'_*}\Big(\frac{\partial w}{\partial h}\,\frac{\partial v}{\partial g} - \frac{\partial w}{\partial g}\,\frac{\partial v}{\partial h}\Big) = \frac{i}{2g^2}\Big[\frac{1}{g}\Big(\frac{\partial w}{\partial h}\Big)^2 + g\,\frac{1+4kg^2}{1+12kg^2}\Big(\frac{\partial w}{\partial g}\Big)^2\Big] \neq 0.$$

Hence, transformation (39) can be inverted: $Z = Z(z,\bar{z})$ and $\bar{Z} = \bar{Z}(z,\bar{z})$. The resultant function $w(Z,\bar{Z})$ can be presented as $w(Z(z,\bar{z}),\bar{Z}(z,\bar{z})) = w(z,\bar{z})$. This function determines the displacement in the physical plane and also the strains and stresses.

We can consider an inverse problem where the complex potential (and the displacement, strains, and stresses determined by this potential) are set, and the load corresponding to a particular problem is to be found.

Example 3. Let $W = 2AZ$, where $A = B + iC$. Then, we have $W' = 2A$, and relations (39), according to (38), have the form

$$z = A\,\mathrm{e}^{Z} - 2k\bar{A}\,\mathrm{e}^{-\bar{Z}}, \qquad \bar{z} = \bar{A}\,\mathrm{e}^{\bar{Z}} - 2kA\,\mathrm{e}^{-Z}.$$

Eliminating $\bar{Z}$ from these relations, we obtain the equation for Z:

$$\bar{z}A\,\mathrm{e}^{2Z} + [2k(A^2-\bar{A}^2) - z\bar{z}]\,\mathrm{e}^{Z} - 2kzA = 0.$$

We find the linear solution with respect to the parameter $k \ll 1$, assuming that $\mathrm{e}^{Z} = f + kg$ ($f \neq 0$). Substituting e^{Z} into the last equation and equating the coefficients at the zeroth and first powers of the parameter to zero, we obtain the equations

$$Af - z = 0, \qquad 2\bar{z}Afg + 2(A^2-\bar{A}^2)f - z\bar{z}g - 2zA = 0$$

determining this approximation. Thus, we have

$$\mathrm{e}^{Z} = \frac{z}{A}\Big(1 + \frac{2k\bar{A}^2}{z\bar{z}}\Big), \qquad \mathrm{e}^{\bar{Z}} = \frac{\bar{z}}{\bar{A}}\Big(1 + \frac{2kA^2}{z\bar{z}}\Big).$$

It follows from here that

$$Z = \ln\frac{z}{A} + \frac{2k\bar{A}^2}{z\bar{z}}, \qquad \bar{Z} = \ln\frac{\bar{z}}{\bar{A}} + \frac{2kA^2}{z\bar{z}}$$

and displacement (35) has the form

$$w = A\ln(\bar{A}z) + \bar{A}\ln(A\bar{z}) - (A+\bar{A})\Big[\ln(A\bar{A}) - \frac{2kA\bar{A}}{z\bar{z}}\Big].$$

Example 4. Let $W = 4F\,\mathrm{e}^{Z}$, where $F = G + iH = \text{const}$. Then, we have $W' = 4F\,\mathrm{e}^{Z}$, and equalities (39), in accordance with Eq. (38) (with $D = 0$), take the form

$$z = F\,\mathrm{e}^{2Z} + 4k\bar{F}\bar{Z}, \qquad \bar{z} = \bar{F}\,\mathrm{e}^{2\bar{Z}} + 4kFZ.$$

Inverting these dependences, we obtain the following relation in the linear approximation with respect to k:

$$Z = \ln\sqrt{\frac{z}{F}} - k\,\frac{\bar{F}}{z}\,\ln\frac{\bar{z}}{\bar{F}}, \qquad \bar{Z} = \ln\sqrt{\frac{\bar{z}}{\bar{F}}} - k\,\frac{F}{\bar{z}}\,\ln\frac{z}{F}.$$

Hence, displacement (35) has the form

$$w = 2[\sqrt{Fz}\,(\bar{F}/\bar{z})^{k\bar{F}/z} + \sqrt{\bar{F}\bar{z}}\,(F/z)^{kF/\bar{z}}].$$

In Examples 3 and 4, the displacement depends on two real parameters.

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